Mathematics: analysis and approaches Higher level
Paper 1

Section A

1. [Maximum mark: 5]

Consider the function $f(x) = 3(x-2)(x-q)$, where $x \in \mathbb{R}$ and q is a real constant.

The axis of symmetry of the graph of f has equation $x = 4$.

(a) Show that $q = 6$. [2]

(b) Find the coordinates of the vertex of the graph of f. [2]

(c) Hence, write down the range of f. [1]

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2. [Maximum mark: 5]

A particle P is travelling along a straight line, with displacement measured relative to a point O on the line. The velocity, v m s^{-1}, of the particle at time t seconds is given by

$$v = \frac{1}{t+1} + e^{-t}$$

where $t \geq 0$.

(a) Find $\int \left(\frac{1}{t+1} + e^{-t}\right) dt$. [2]

Initially the particle is at O.

(b) Find an expression for the displacement, s metres, in terms of t. [3]

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3. [Maximum mark: 6]

Consider a geometric sequence, u_n, with common ratio r, where each term in the sequence is positive.

It is given that $u_6 = 1.6 \times 10^5$ and $u_8 = 6.4 \times 10^5$.

(a) Find the value of r. [3]

(b) Find the value of u_1, giving your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}$. [3]

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4. [Maximum mark: 7]

(a) Show that $\log_{25}(81x^2) = \log_5 9 + \log_5 x$ for $x > 0$. [4]

(b) Hence, solve the equation $\log_5 \sqrt[3]{x} = \log_{25}(81x^2) - \log_5 4$ where $x > 0$. [3]

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5. [Maximum mark: 9]

Consider two events, A and B.

It is given that $P(A' \cap B) = \frac{1}{6}$, $P(A \cap B) = \frac{7-x}{12}$ and $P(A \cap B') = \frac{5+x}{24}$, where $0 \leq x \leq 7$.

(a) Show that $P(A) = \frac{19-x}{24}$. [3]

It is given that A and B are independent.

(b) Find the possible values of x. [6]

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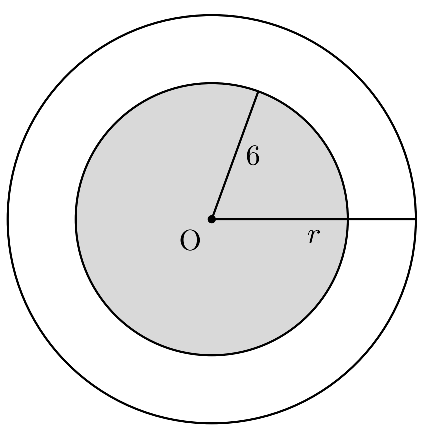
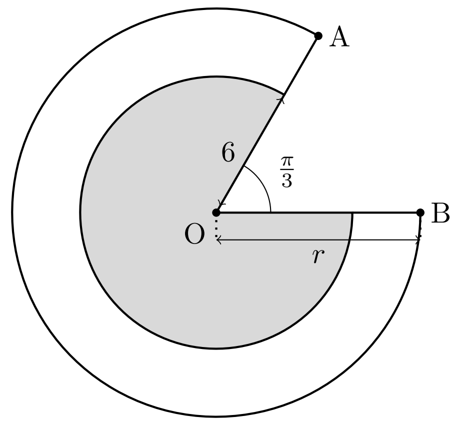
6. [Maximum mark: 4]

Consider a piece of paper in the shape of a circle with centre O and radius r cm, where $r > 6$. A second circle with centre O and radius 6 cm is drawn inside the first and shaded, as shown in the following diagram.



The points A and B are on the circumference of the larger circle such that the acute angle $A\hat{O}B = \frac{\pi}{3}$. The paper is cut along the lines AO and BO and the sector AOB with a central angle of $\frac{\pi}{3}$ is removed as shown on the following diagram.



After removing the sector, it is now given that

$$\frac{\text{the area of the unshaded region}}{\text{the area of the shaded region}} = \frac{2}{3}$$

Find the value of r, giving your answer in the form $\sqrt{k}$, where $k \in \mathbb{Z}$.

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7. [Maximum mark: 6]

Reyn is investigating the fertility of a group of adult female foxes. He has developed a clinical test that aims to identify whether or not a given fox is pregnant, and he wants to evaluate the reliability of this test.

Reyn knows that

- 40% of the foxes in the group are pregnant
- if a fox is pregnant, the probability that it will test positive is 90%
- if a fox is not pregnant, the probability it will test positive is 20%.

(a) Find the probability that a randomly selected fox from the group will test positive. [3]

(b) Find the probability that a randomly selected fox from the group is pregnant, given that it tested positive. [3]

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8. [Maximum mark: 8]

Consider the polynomial $p(z) = 3z^5 - 7z^4 + cz^3 + dz^2 + ez - 26$ where $z \in \mathbb{C}$ and $c, d, e \in \mathbb{R}$.

Three of the roots of $p(z)$ are $\frac{1}{3}$, $a - i$, $2 + bi$, where $a, b \in \mathbb{R}$, $b > 1$.

(a) By considering the sum and the product of the roots of $p(z)$,

(i) show that $a = -1$;
(ii) find the value of b. [5]

(b) Show that $p(z) = (Az + B)(z^2 + Cz + D)(z^2 + Ez + F)$, where A, B, C, D, E, F are integers to be determined. [3]

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9. [Maximum mark: 8]

Consider non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in three-dimensional space.

The non-zero vector $\mathbf{v}$ is defined as $\mathbf{v} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

(a) Show that $\mathbf{v} \cdot \mathbf{a} = 0$. [2]

It is given that $\mathbf{b} \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ are non-zero vectors.

(b) (i) Justify why $\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

(ii) Show that $\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c}) = 0$. [4]

(c) Hence, state the geometrical relationship between $\mathbf{v}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, briefly justifying your answer. [2]

Section B

10. [Maximum mark: 15]

Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where $x \in \mathbb{R}$ and where a, b, c and d are real constants.

The graph of f has a point of inflexion at $(-2, -2c + d + 24)$.

(a) Show that $a = \frac{3}{2}$ and $b = 9$. [6]

It is given that the tangents to the graph of f at $x = -3$ and $x = k$ are horizontal.

(b) (i) Show that $c = \frac{27}{2}$.
(ii) Find the value of k.
(iii) State whether f has a local maximum or a local minimum at $x = k$, justifying your answer. [7]

The graph of f intersects the y-axis at the point P.

(c) Show that the tangent to the graph of f at $x = -3$ passes through P. [2]

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11. [Maximum mark: 19]

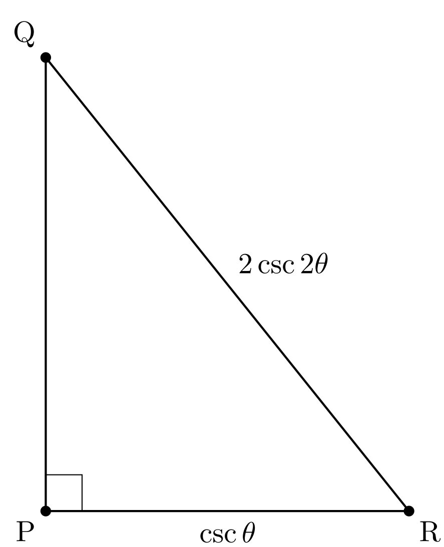
(a) Prove the identity $\cot 2\theta \equiv \frac{1}{2}\left(\cot\theta - \frac{1}{\cot\theta}\right)$. [3]

(b) Hence, solve the equation $\cot 2\theta = \frac{3}{2}\cot\theta(\cot^2\theta - 1)$ for $0 < \theta < \pi$ where $\theta \neq \frac{\pi}{2}$. [6]

(c) By differentiating both sides of the identity in part (a), show that $4\csc^2 2\theta \equiv \csc^2\theta + \sec^2\theta$. [5]

Consider the right-angled triangle ΔPQR, where $R\hat{P}Q = \frac{\pi}{2}$, $PR = \csc\theta$ and $QR = 2\csc 2\theta$, where $0 < \theta < \frac{\pi}{2}$ and all lengths are given in metres.

This is shown in the following diagram.



QR has length L metres and ΔPQR has area A square metres.

(d) Calculate the ratio $L : A$ in the form $n : 1$ where $n \in \mathbb{Z}^+$. [5]

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12. [Maximum mark: 18]

Rowan is investigating the spread of an infection through a population of rabbits. He models N, the number (in thousands) of infected rabbits in the population, at time t days.

Rowan models the spread of the infection by the differential equation

$$\frac{dN}{dt} = k(-3 + 4N - N^2)$$

where $t \geq 0$ and $k \in \mathbb{R}^+$.

Initially, 1500 rabbits are infected.

(a) Express $\frac{1}{-3+4x-x^2}$ in the form $\frac{A}{x-1} + \frac{B}{3-x}$, where $A, B \in \mathbb{R}$. [3]

(b) Hence, show that $\ln\left(\frac{N-1}{3-N}\right) = 2kt - \ln p$, where $p \in \mathbb{R}^+$ is a constant to be determined. [8]

(c) Find an expression for N in the form $N = \frac{a+be^{-2kt}}{1+ce^{-2kt}}$ where $a, b, c \in \mathbb{Z}$. [5]

According to Rowan's model, the number of infected rabbits approaches a limit, L, over the long term.

(d) Find the value of L. [2]